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On the running electromagnetic coupling constant at M_Z

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Abstract. We present a discussion on how to define the running electromagnetic coupling constant at M_Z or some other intermediate scale, e.g., m_Υ. We argue that a natural definition consistent with general requirements of the renormalization group should use Euclidean values of the momentum of the photon propagator as the appropriate scale. We demonstrate explicitly, through the evaluation of the running coupling constant at the scale of the Υ-resonance mass, that the usual definition of the hadronic contribution with a principal-value prescription is inconsistent. In the determination of the value of α at M_Z, we use a Euclidean definition rather than the principal-value one, and as a result, the numerical difference is comparable in size to the errors caused by existing experimental and QCD inputs to the evaluation of $\alpha(M_Z)$.

In applications to high precision tests of the standard model [1] with observables near the Z-boson peak, the electromagnetic coupling constant should be used at a scale of the order of the Z-boson mass M_Z, (see, e.g., [2,3]). The running electromagnetic coupling constant at M_Z has even been chosen as a standard reference parameter [4]. It differs numerically from the value of the fine structure constant $\alpha^{-1} = 137.036\ldots$ defined at zero momentum, and from that in Coulomb's law for heavy nonrelativistic particles. The change is usually accounted for through the renormalization group equation [5,6]. Because the fine structure constant is defined at vanishing momentum, and is taken as the initial value in the solution of the renormalization group equation, the running electromagnetic coupling constant at M_Z is an infrared-sensitive quantity inasmuch as the strong interaction contributions are not easy to compute, because the region is nonperturbative at small energies. Therefore this contribution is usually taken into account in the leading order of electromagnetic interaction, within a semiphenomenological approximation through a dispersion relation. There has been a renewal of interest in a precise determination of the hadronic contribution during recent years in particular in connection with the constraints on the Higgs-boson mass [7]. Some recent references giving a state-of-the-art analysis of this contribution are [8–11]. A quasi-analytical approach is used in [12], where some references to earlier papers can be found (see also [13,14]). An extremely thorough data-based analysis is given in [15].

In the present paper, we critically discuss the definition of the running electromagnetic coupling constant at M_Z as it is used in the literature. The standard approach consists in using the principal-value prescription at the appropriate scale in the physical domain on the positive energy semiaxis. We argue that a natural definition, that is consistent with general requirements and the standard notion of running used in renormalization group applications should use the Euclidean momentum of the photon propagator as the appropriate scale.

The running coupling constant $\alpha(q^2)$ is defined through the (one-photon, irreducible) photon vacuum polarization function $\Pi_\gamma(q^2)$ as

$$\alpha(q^2) = \frac{\alpha}{1 - \Pi_\gamma(q^2)}. \tag{1}$$

$\Pi_\gamma(q^2)$ contains both leptonic and hadronic contributions. The hadronic part of the polarization function $\Pi_\gamma(q^2)$ (with one subtraction at zero momentum) reads

$$\Pi_\gamma^{\text{had}}(q^2) = -\frac{\alpha}{3\pi} q^2 \int\limits_{4m_\pi^2}^{\infty} \frac{R_h(s)\mathrm{d}s}{s(s - q^2 - \mathrm{i}0)} \tag{2}$$

where $R_h(s)$ is the normalized cross section of e^+e^- annihilation into hadrons. Let us introduce, for convenience, the polarization function $\Pi(q^2)$,

$$\Pi(q^2) = -q^2 \int_0^\infty \frac{R(s)\mathrm{d}s}{s(s - q^2 - \mathrm{i}0)} \tag{3}$$

such that

$$\Pi_\gamma(q^2) = \frac{\alpha}{3\pi}\Pi(q^2) \tag{4}$$

where $R(s)$ is the corresponding spectral density. Note that $\alpha(q^2)$ is defined for every complex value of q^2 by (2). For real, negative q^2, the polarization function $\Pi_\gamma(q^2)$ (and $\Pi(q^2)$ as well) is a positive real number, because the spectral density $R(s)$ is positive.

The definition (1) is used in renormalization group applications, and the scale q^2 is taken to be a negative real number that corresponds to a propagator in the Euclidean domain. The Euclidean definition is usually used in applications of grand unified theories [16], supersymmetry at large energy [17], and physics at the Planck scale.

For the precise study of the physics at the Z-boson pole, the effective electromagnetic interaction coupling constant $\bar{\alpha}$ is represented in the form

$$\bar{\alpha} = \frac{\alpha}{1 - \Delta\alpha}\,. \tag{5}$$

Numerically one obtains a positive real number for $\Delta\alpha$.

In the present literature, a theoretical expression for $\Delta\alpha$ is defined directly on the positive semiaxis, by the use of the principal-value prescription for the singularity of the integrand in (2):

$$\Delta\alpha = \mathrm{Re}\,\Pi_\gamma(M_Z^2),$$
$$\mathrm{Re}\,\Pi_\gamma(M_Z^2) = -\frac{\alpha}{3\pi} M_Z^2\, \mathrm{P}\int_{4m_\pi^2}^{\infty} \frac{R(s)\mathrm{d}s}{s(s - M_Z^2)}. \tag{6}$$

Here $\mathrm{P}\int$ denotes the principal value of the integral. This makes $\Delta\alpha$ real (the initial i0 prescription for the integral gives it an imaginary part), which is appropriate for a coupling constant. We argue that this prescription is not adequate for the physical situation at hand and does not correspond to the notion of a running coupling used in the standard renormalization group applications. The latter corresponds to scales taken in the Euclidean domain

$$\alpha_\mathrm{E}(\mu^2) = \frac{\alpha}{1 - \Pi_\gamma(-\mu^2)},$$
$$\Pi_\gamma(-\mu^2) = \frac{\alpha}{3\pi}\, \mu^2 \int_{4m_\pi^2}^{\infty} \frac{R(s)\mathrm{d}s}{s(s + \mu^2)}. \tag{7}$$

The running electromagnetic coupling constant at the scale M_Z is then defined as $\alpha_\mathrm{E}(M_Z^2)$. Therefore, for the phenomenological parameter "running electromagnetic coupling constant at the scale M_Z", denoted by $\bar{\alpha}$, we have two representations:

i) the standard one with a principal-value prescription,

$$\bar{\alpha} = \alpha_\mathrm{PV}(M_Z^2), \qquad \Delta\alpha = \mathrm{Re}\,\Pi_\gamma(M_Z^2); \quad \text{and} \tag{8}$$

ii) the alternative one in the Euclidean domain,

$$\bar{\alpha} = \alpha_\mathrm{E}(M_Z^2), \qquad \Delta\alpha = \Pi_\gamma(-M_Z^2), \tag{9}$$

which is defined through

$$\alpha_\mathrm{E}(M_Z^2) = \frac{\alpha}{1 - \Pi_\gamma(-M_Z^2)},$$
$$\Pi_\gamma(-M_Z^2) = \frac{\alpha}{3\pi}\, M_Z^2 \int_{4m_\pi^2}^{\infty} \frac{R(s)\mathrm{d}s}{s(s + M_Z^2)}. \tag{10}$$

We suggest that the Euclidean version be used. The idea of changing scales is embodied in the renormalization group equation, which allows one to control large logarithms. Therefore, theoretically, one is dealing with a logarithm of the ratio of two scales. Note that the notion of scale becomes rather imprecise as soon as complex numbers are involved. For example, numerically M_Z^2 has the same "scale" as $\mathrm{e}^{\mathrm{i}\pi} M_Z^2 = -M_Z^2$. The choice of an appropriate scale is determined by the particular kinematics and the higher order corrections of each process in question. In the leading logarithmic approximation, however, keeping the finite corrections to large logarithms is beyond the accuracy of an approximation that renders all scales with the same absolute value equivalent. As a reference value for the coupling constant the usual choice is to take a Euclidean point. This problem is discussed for strong interactions in [18], where different versions of a real part and an absolute value definition of the coupling constant at complex points have been also studied. In the general case, it is difficult to decide on how to deal with observables, including complex numbers within the renormalization group resummation of logarithms. For two-point functions, however, there is a natural solution to this problem, based on their analytic properties: the dispersion representation [19,20]. Below, we discuss these two possibilities of defining the running electromagnetic coupling constant at the scale M_Z.

First we show that the two (Euclidean and principal-value) definitions are numerically close for applications in the vicinity of the Z-boson peak discussed in the literature. Let us take (2) and split the whole region of integration into two parts separated by s_0:

$$\Pi(q^2) = -q^2 \int_0^{s_0} \frac{R(s)\mathrm{d}s}{s(s - q^2 - \mathrm{i}0)} - q^2 \int_{s_0}^{\infty} \frac{R(s)\mathrm{d}s}{s(s - q^2 - \mathrm{i}0)}. \tag{11}$$

If $|q^2|$ is chosen such that $|q^2| \gg s_0$, one can expand the denominator in the first integral. Then, if s_0 is large enough, one can use perturbation theory for the spectral density in the second integral. For illustrative purposes, we choose a very simplified approximation for $R(s)$, namely $R(s) = \mathrm{const} = 1$ for $s > s_0$, and we obtain

$$\Pi(q^2) = \int_0^{s_0} \frac{R(s)}{s}\mathrm{d}s + \ln\frac{|s_0 - q^2|}{s_0}, \tag{12}$$

where the principal-value prescription has been used. Expanding (12) in the limit $|q^2| \gg s_0$, we finally obtain

$$\Pi(q^2) = \int_0^{s_0} \frac{R(s)}{s}\mathrm{d}s + \ln\frac{|q^2|}{s_0}, \tag{13}$$

which is independent of the phase of the complex number q^2. The same result can be obtained directly from (3) in this limit. Therefore, in the above approximation, with the suggested regime of variables, the Euclidean and principal-value definitions are equivalent numerically. Later on, we discuss corrections to this leading order approximation that depend on whether the Euclidean or principal-value definition is used.

This is a qualitative picture. Because the above simplifying assumptions can be expected to correctly embody the main features of a more sophisticated numerical analysis, it is clear that the numerical change, which stems from our choosing q^2 in the Euclidean domain rather than using the conventional definition, is under control at the scale of M_Z, and does not jeopardize current phenomenology. However, the definition in the Euclidean domain given in (10) is preferable from a theoretical point of view: It is natural; it gives a real number; it is smooth; and it is consistent with the renormalization group.

The principal-value prescription has equally obvious deficiencies. It is *ad hoc*; it gives a real part of a propagator which is not directly related to a coupling constant in a renormalization group sense; and it is not smooth.

The last deficiency listed is, in fact, the most crucial one. Let us present more details. We take the principal-value definition at $q^2 = M_Z^2$ and compute the polarization function for a model spectral density $R(s) = \theta(s - s_0)$, obtaining

$$\Pi(M_Z^2) = \ln \frac{|s_0 - M_Z^2|}{s_0} . \tag{14}$$

Note first that the polarization function (14) gives a rather curious result,

$$\mathrm{P}\int_{M_Z^2/2}^{\infty} \frac{\mathrm{d}s}{s(s - M_Z^2)} = 0 \tag{15}$$

which means that the contributions of all states with masses larger than $M_Z/\sqrt{2} \sim 60$ GeV are exactly equal to zero, assuming that the asymptotic spectral density in this region is constant. Also, there is a sign change in the vicinity of $M_Z/\sqrt{2}$. This feature persists for any realistic $R(s)$ in the vicinity of some point $s^* \simeq M_Z^2/2$, because in this region, QCD perturbation theory works well, and the spectral density is rather smooth and close to its asymptotic value, which is almost a constant (up to a slow logarithmic decrease). Therefore one gets the exact equality

$$\mathrm{P}\int_{s^*}^{\infty} \frac{R(s)\mathrm{d}s}{s(s - M_Z^2)} = 0 \tag{16}$$

for some $s^* \sim M_Z^2/2$.

Furthermore, if one takes $s_0 = M_Z^2$ in (14) then the logarithm is ill-defined. It can be seen that this is a consequence of the principal-value definition (6). Though this is a rather academic example, we nevertheless take it as a warning, because there is no sharp increase of the spectral density, nor changes, in general, in the vicinity of the Z-boson mass. More realistic situations are considered below. In contrast to the principal-value definition, the Euclidean definition is also fine in this case:

$$\Pi(-M_Z^2) = \ln \frac{s_0 + M_Z^2}{s_0} . \tag{17}$$

The reason for the ill-defined behavior of (14) and (6) is clear. The principal-value prescription leads to a distribution $\mathrm{P}1/x$ which is defined only on smooth functions. A product of two distributions,

$$\left(\mathrm{P}\frac{1}{s - M_Z^2}\right)\theta(s - M_Z^2) \tag{18}$$

is not an integrable function. An *ad hoc* definition with a principal-value prescription fails to define a value for the running coupling at some particular points and particular functions $R(s)$ in (6); thus one has to introduce further rules for such cases.

Additionally, in a more realistic situation, one needs the running electromagnetic coupling constant at the scales around the J/ψ- or Υ-family resonance masses in order to account for their leptonic widths. With the principal-value definition, it is impossible to compute the running electromagnetic coupling constant at the scales around the resonance masses. Indeed, the correction $\Delta\alpha$ to the running electromagnetic coupling constant at the scale of the Υ-meson mass, m_Υ, is given by an ill-defined integral of the product of two distributions:

$$\left(\mathrm{P}\frac{1}{s - m_\Upsilon^2}\right)\delta(s - m_\Upsilon^2). \tag{19}$$

This quantity is not defined as a distribution because a product of two distributions is not defined when their singular points coincide.

Leaving the mathematical statement about the ill-defined behavior of a product of two distributions aside, in practice, the results following from the principal-value definition will be unstable (for finite widths of the resonances or for some sharp, but still smooth, increase of the spectral function). As an explicit example, we take the spectral density corresponding to a single Breit–Wigner resonance and calculate its contribution to $Re\Pi(M_Z^2)$. The Breit–Wigner spectral function is given by

$$R_{\mathrm{BW}}(s) = \frac{1}{\pi}\frac{\Gamma M}{(s - M^2)^2 + \Gamma^2 M^2}, \tag{20}$$

where M and Γ are the mass and the width of the resonance, respectively. As $\Gamma \to 0$, one obtains $R_{\mathrm{BW}}(s) \to \delta(s - M^2)$. After integration (with proper care taken for the point $s = 0$), one finds

$$\begin{aligned} M_Z^2\mathrm{P}\int^{\infty} \frac{R_{\mathrm{BW}}(s)\mathrm{d}s}{s(s - M_Z^2)} &= \frac{M^2 - M_Z^2}{(M^2 - M_Z^2)^2 + \Gamma^2 M^2} + \dots \\ &= \frac{\Delta}{\Delta^2 + \Gamma^2 M^2} + \dots, \end{aligned} \tag{21}$$

with $\Delta = M^2 - M_Z^2$. Only potentially singular terms for the limit $\Gamma \to 0$, in the vicinity $M^2 \sim M_Z^2$ have been kept. The last function in (21) has its extremal points at $\Delta = \pm\Gamma M$, with the values

$$\left.\frac{\Delta}{\Delta^2 + \Gamma^2 M^2}\right|_{\Delta = \pm\Gamma M} = \pm\frac{1}{2\Gamma M}. \tag{22}$$

There is no regular limit $\Gamma \to 0$, and further rules would be required to deal with this limit. Note that the result (21) can be obtained without explicit integration of the Breit–Wigner spectrum. Since the Breit–Wigner function can be regarded as a regularization of a δ-distribution, one can introduce a regularization of $\mathrm{P}1/x$ in the class of infinitely smooth functions instead. For example,

$$\lim_{\epsilon\to 0}\frac{x}{x^2+\epsilon^2} = \mathrm{P}\frac{1}{x} \qquad \text{and}$$
$$\lim_{\epsilon\to 0}\frac{(s-M_Z^2)}{(s-M_Z^2)^2+\epsilon^2} = \mathrm{P}\frac{1}{s-M_Z^2}\,.$$

Then after the integration with an infinitely narrow resonance one gets

$$M_Z^2\int\frac{\delta(s-M^2)(s-M_Z^2)\mathrm{d}s}{s[(s-M_Z^2)^2+\epsilon^2]} = \frac{M_Z^2}{M^2}\frac{\Delta}{\Delta^2+\epsilon^2} = \frac{\Delta}{\Delta^2+\epsilon^2}+\dots \quad (23)$$

for $M^2 \sim M_Z^2$. The regularization cannot be unambiguously removed, i.e., there is no unique limit at $\epsilon = 0$ in the vicinity of $\Delta = 0$. Of course, this is a reflection of the fact that the product of two distributions is ill-defined.

We will not dwell on the ill-defined behavior that results when a θ-function-type spectral density is used. This is a realistic possibility when one computes the light-quark contributions to the running electromagnetic coupling constant that is normalized in the vicinity of a sharp rise of the spectral density around 1.5 GeV^2.

With a Euclidean definition none of the above difficulties appears. Also, because in this case the polarization function is defined in the Euclidean domain, one need not integrate all data (only a small region near the origin requires explicit integration). Thus we can use all the power of perturbation theory (e.g., [21]), although special care has to be taken with regard to the subtraction at zero momentum that enters the definition of the coupling constant and makes it an infrared sensitive quantity. For this purpose, however, more sophisticated means can be used that will increase the accuracy [22–24].

Two further remarks are called for here. The first concerns the leptonic contribution. For a lepton with the mass m_l, the asymptotic form of this contribution reads

$$\Pi_{\mathrm{lept}} = \ln\left(\frac{M_Z^2}{m_l^2}\right) - \frac{5}{3}, \quad (24)$$

and has the same real part for any real phase φ of $\mathrm{e}^{\mathrm{i}\varphi}M_Z^2$. When the asymptotic form is used, both prescriptions are equivalent numerically.

The second remark concerns the higher order contributions of the electroweak interactions. In the next order of the electroweak interaction, there is a contribution of the Z-boson peak to the polarization function $\Pi_\gamma(q^2)$, due to γZ transitions (e.g., [25]). Therefore, in that order, one has to interpret the product of the principal-value distribution with the sharp Breit–Wigner spectrum of the Z-boson pole itself. Again, this problem is not present within the Euclidean definition.

Now we briefly discuss the numerical difference that can result from the change of the definition of the running coupling constant at M_Z. Our model for the hadronic spectral density $R(s)$ is simple, and is mainly designed for illustrative purposes, so that one can easily trace the difference between the principal-value and Euclidean definitions of the running coupling constant. The spectral density is chosen such that all calculations can be done analytically; this is convenient for estimating the order of magnitude of the difference between the two definitions. For the light quarks u, d, s we assume the existence of a low-lying resonance (e.g., ρ, ω, and φ) and a continuum. In general, we take the following form of the spectrum for every light-quark flavor:

$$R_{\mathrm{light}}(s) = 3Q_q^2[2m^2\delta(s-m^2)+\theta(s-2m^2)],$$

according to the model of [26] with Q_q being a light-quark fractional charge. The couplings of the low-lying resonances have been replaced by the duality interval $2m^2$. For heavy quarks, we take the simplest model of the form

$$R_{\mathrm{heavy}}(s) = 3Q_Q^2\theta(s-4m_Q^2),$$

which represents the partonic asymptotic value, with a naive step function two-quark-threshold. Q_Q is a heavy-quark charge. Collecting everything together, we find the following results. The three light quarks give the result for $\Pi(q^2)$ in the general form:

$$\Pi_{\mathrm{light}}(q^2) = 2\left(\frac{-2q^2}{m^2-q^2-\mathrm{i}0} + \ln\frac{|2m^2-q^2|}{2m^2}\right).$$

Expanding at $|q^2| \sim M_Z^2$, one gets

$$\Pi_{\mathrm{light}} = 2\left[2+\ln\frac{M_Z^2}{2m^2}+O(m^6/M_Z^6)\right] \quad (25)$$

which gives the same answer for both definitions, with high precision. The difference starts at order $O(m^6/M_Z^6)$, and is completely negligible for light-resonance masses of order 1 GeV (ρ , ω , and φ resonances, for instance). For the charm contribution, we find

$$\Pi_{\mathrm{charm}}^{\mathrm{PV}} = \frac{4}{3}\ln\frac{M_Z^2-4m_c^2}{4m_c^2}$$

in the case of the principal-value prescription. In the case of the Euclidean prescription, one has

$$\Pi_{\mathrm{charm}}^{\mathrm{E}} = \frac{4}{3}\ln\frac{M_Z^2+4m_c^2}{4m_c^2}\,.$$

Expanding these formulas in the small ratio $4m_c^2/M_Z^2$, one finds, in the leading order, the following representation:

$$\Pi_{\mathrm{charm}}^{\mathrm{PV,E}} = \frac{4}{3}\ln\frac{M_Z^2}{4m_c^2}\left(1\mp\frac{4m_c^2}{M_Z^2}\frac{1}{\ln\frac{M_Z^2}{4m_c^2}}\right).$$

We keep this form for further numerical comparisons to be made in the case of b and t quarks. For the b-quark contribution, one gets

$$\Pi^{\mathrm{PV,E}}_{\mathrm{bottom}} = \frac{1}{3}\ln\frac{M_Z^2}{4m_b^2}\left(1 \mp \frac{4m_b^2}{M_Z^2}\frac{1}{\ln\frac{M_Z^2}{4m_b^2}}\right).$$

For the t-quark contribution, the result reads

$$\Pi^{\mathrm{PV,E}}_{\mathrm{top}} = \frac{4}{3}\ln\left(1 \mp \frac{M_Z^2}{4m_t^2}\right),$$

and the opposite limit $M_Z \ll 2m_t$ can be used to evaluate this contribution for two different definitions with the necessary accuracy. For numerical estimates of the corresponding contributions, we take $\sqrt{2}m = 1$ GeV for the light-quark resonances, $m_c = 1.4$ GeV, $m_b = 4.8$ GeV, $m_t = 175$ GeV, and $M_Z = 91$ GeV. Note that the exact definition of the quark-mass parameters is not required here because it is far beyond the accuracy of our simple model. Moreover, these parameters can be considered as effective parameters for describing integrals over the threshold regions of quark production. Nevertheless, we stay close to canonical values for the quark pole mass. While the absolute value of the contribution will be obtained rather approximately, the model is nevertheless sufficient for our main purpose: to estimate the difference between the two definitions. Numerically, (25) leads to the light-quark contribution

$$\text{light quarks}(u,\, d,\, s) = 2 \times (2 + 2 \times 4.5) = 22.0$$

with both prescriptions. For the contributions of the heavy quarks, we find

$$\begin{aligned} c\text{ quark} &= \frac{4}{3} \times 7.0(1. \mp 0.14 \times 10^{-3}) \\ b\text{ quark} &= \frac{1}{3} \times 4.5(1. \mp 2.5 \times 10^{-3}) \\ t\text{ quark} &= \frac{4}{3} \times (\mp 68 \times 10^{-3}). \end{aligned} \tag{26}$$

Summing everything together, one obtains the total quark contribution to the hadronic vacuum polarization in the form

$$\begin{aligned} &22.0 + \frac{4}{3} \times 7.0(1. \mp 0.14 \times 10^{-3})_c \\ &\quad + \frac{1}{3} \times 4.5(1. \mp 2.5 \times 10^{-3})_b \\ &\quad + \frac{4}{3} \times (\mp 68 \times 10^{-3})_t \\ &\quad = 32.8 + 0.1\,\delta \approx 33 + 0.1\,\delta, \end{aligned} \tag{27}$$

where $\delta = -1$ for the principal-value definition, and $\delta = 1$ for the Euclidean definition. One sees that the difference is saturated by the top-quark contribution. This is natural, because the mass of a top quark is closest to M_Z. Its contribution to the hadronic vacuum polarization is small in absolute value, but is completely different for the two definitions. In the standard notation, we have from (27),

$$\Delta\alpha_{\mathrm{had}} = \frac{\alpha}{3\pi}\,(33 + 0.1\,\delta)\,, \tag{28}$$

with $\alpha^{-1} = 137.036$ being the fine structure constant. The lepton contribution is taken into account according to the asymptotic formula (24), with the following numerical values of lepton (e,μ,τ) masses: $m_e = 0.5$ MeV, $m_\mu = 0.1$ GeV, and $m_\tau = 1.8$ GeV. This gives

$$(24.2)_e + (13.6)_\mu + (7.8)_\tau - 5 = 40.6 \approx 41\,.$$

In the standard notation, the lepton contribution reads

$$\Delta\alpha_{\mathrm{lep}} = \frac{\alpha}{3\pi}\,41. \tag{29}$$

The final result for the total contribution of the charged fermions to the vacuum polarization function becomes

$$33 + 0.1\,\delta + 41 = 74 + 0.1\,\delta,$$

which, in the standard notation, reads

$$\Delta\alpha = \frac{\alpha}{3\pi}\,(74 + 0.1\,\delta) = \alpha(7.9 + 0.01\,\delta).$$

For the inverse running electromagnetic coupling constant at the scale M_Z, we obtain

$$\alpha^{-1}(M_Z) = 137.0 - (7.9 + 0.01\,\delta) = 129.1 - 0.01\,\delta\,.$$

Even though the central value of our approximate evaluation is rather close to the results of more precise evaluations [8–11] this agreement should not be taken too seriously. Our estimate is rather rough and serves the purpose of obtaining the numerical change between the two definitions of the running electromagnetic coupling constant at M_Z. At present, this change has no decisive influence on current phenomenology. It is within the error bars of uncertainty for the more precise values 128.93 ± 0.06 [9] or even the smaller errors $128.93 \pm 0.015_{\mathrm{exp}} \pm 0.015_{\mathrm{th}}$ [8, 10, 11]. Manifestly, taking the recent results of evaluation with the principal-value definition from [11] (for the sake of normalization),

$$\Delta\alpha_{\mathrm{had}}(\text{ref. [11]}) = (276.3 \pm 1.6) \times 10^{-4},$$

we have to move the mean value by 1.5×10^{-4} due to the change of definition (27). This leads to

$$\Delta\alpha_{\mathrm{had}}(\text{Euclidean}) = (277.8 \pm 1.6) \times 10^{-4}\,.$$

Correspondingly, instead of the value

$$\alpha^{-1}(M_Z)_{\mathrm{PV}} = 128.933 \pm 0.021$$

of [11], we have

$$\alpha^{-1}(M_Z)_{\mathrm{Euclidean}} = 128.913 \pm 0.021.$$

In the last formula, we have neglected the change caused by the redefinition of the leptonic contribution. Therefore the Euclidean definition, which we consider more consistent theoretically, does not violate current phenomenology.

However, the change of the mean numerical value of the inverse running electromagnetic coupling constant at M_Z, -0.02, is comparable in size with the present uncertainty, 0.021. In the future, when the experimental data used in the determination of the running electromagnetic coupling constant at M_Z improves, the difference between the two definitions will become significant.

Our last numerical example concerns the order of magnitude of the singular term in the coupling normalized at the scale m_Υ, within the principal-value prescription. With the same normalization as in our simple model, the contribution of the Υ resonance to the spectral density in the Breit–Wigner approximation reads

$$R_\Upsilon(s) = \frac{2}{3} m_\Upsilon \Delta_\Upsilon R_{\mathrm{BW}}(s, m_\Upsilon, \Gamma), \qquad (30)$$

where $\Delta_\Upsilon \approx 1$ GeV is its duality interval linear in energy [27] related to its leptonic decay width, $\Delta_\Upsilon = 27\pi\Gamma(\Upsilon \to e^+e^-)/2\alpha^2$ with $\Gamma(\Upsilon \to e^+e^-) = 1.32$ keV, while $\Gamma = 52.5$ keV is its full width. The singular contribution of the Υ resonance to the polarization function abruptly changes between the two extremes taken from (22) as the normalization point passes the position of the resonance:

$$\pm \frac{2}{3} m_\Upsilon \Delta_\Upsilon \left(\frac{1}{2\Gamma m_\Upsilon} \right)$$

Numerically, one gets

$$\frac{\Delta_\Upsilon}{3\Gamma} = 6.7 \times 10^3,$$

which is far too big from the point of view of phenomenology.

The examples presented in this paper clearly demonstrate the inconsistency of the present definition of the running electromagnetic coupling constant within the principal-value prescription. However, these problems are not noticeable when one discusses a normalization point around M_Z, because the hadronic spectral density is smooth in this region. We will present the results of an accurate numerical analysis within the Euclidean definition in another paper.

To conclude, we suggest that the Euclidean definition of the running electromagnetic coupling constant at M_Z be used as a reference parameter for high precision tests of the standard model at the Z-boson peak. It is free of the shortcomings of the present definition, which is based on the propagator at the physical value of the Z-boson mass, within the principal-value prescription.

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